Elementary maths for GMT

Linear Algebra

Part 2: Matrices, Elimination and Determinant

$m \times n$ matrices

- The system of m linear equations in n variables $x_1, x_2, \ldots, x_n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{cases}$$

can be written as the matrix equation $Ax = b$, i.e.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$m \times n$ matrices

- The matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

has m rows and n columns, and is called a $m \times n$ matrix

Special matrices

- A square matrix (for which $m = n$) is called diagonal matrix if all elements a_{ij} for which $i \neq j$ are zero
- If all elements a_{ii} are one, then the matrix is called the identity matrix, denoted with I_m
  - depending on the context, the subscript m may be omitted

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- If all matrix entries are zero, then the matrix is called zero matrix or null matrix, denoted with 0

Matrix addition

- For two matrices A and B, we have $A + B = C$, with $c_{ij} = a_{ij} + b_{ij}$
  - For example

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 14 \\ 10 & 16 \\ 12 & 18 \end{bmatrix}$$

- Q: What are the conditions on the dimensions of the matrices A and B?

Matrix multiplication

- Multiplying a matrix with a scalar is defined as follows: $cA = B$ with $b_{ij} = ca_{ij}$
  - For example

$$2\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = \begin{bmatrix}2 & 4 & 6\\8 & 10 & 12\\14 & 16 & 18\end{bmatrix}$$

Matrix multiplication

- Multiplying two matrices is a bit more involved
- We have $AB = C$ with $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$
  - For example

$$\begin{bmatrix} 6 & 5 & 1 & -3 \\ -2 & 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 2 \\ 37 & 5 & 16 \end{bmatrix}$$

- Q: What are the conditions on the dimensions of the matrices A and B? What are the dimensions of C?

Properties of matrix multiplication

- Matrix multiplication is associative and distributive over addition

$$(AB)C = A(BC)$$
$$A(B+C) = AB + AC$$
$$(A+B)C = AC + BC$$

- However, matrix multiplication is not commutative, in general, $AB \neq BA$

- Also, if $AB = AC$, it does not necessarily follow that $B = C$ (even if A is not the zero matrix)

Zero and identity matrix

- The zero matrix 0 has the property that if you add it to another matrix A, you get precisely A again

$$A + 0 = 0 + A = A$$

- The identity matrix I has the property that if you multiply it with another matrix A, you get precisely A again

$$AI = IA = A$$

Matrix and 2D linear transformation

- The matrix multiplication of a 2×2 square matrix and a 2×1 matrix gives a new 2×1 matrix
  - For example

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- We can interpret a 2×1 matrix as a 2D vector or point; the 2×2 matrix transforms any vector (or point) into another vector (or point)
  - More later...

Transposed matrices

- The transpose A^T of a $m \times n$ matrix A is a $n \times m$ matrix that is obtained by interchanging the rows and columns of A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Transposed matrices

- For example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- For the transpose of the product of two matrices we have

$$(AB)^T = B^T A^T$$

  – Note the change of order

The dot product revisited

- If we regard (column) vectors as matrices, we see that the dot product of two vectors can be written as $u \cdot v = u^T v$
  – For example

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix} = [32] = 32$$

- A 1×1 matrix is simply a number, and the brackets are omitted

Inverse matrices

- The inverse of a matrix A is a matrix A^{-1} such that

$$AA^{-1} = I$$

- Only **square** matrices **possibly** have an inverse

- Note that the inverse of A^{-1} is A, so we have

$$AA^{-1} = A^{-1}A = I$$

Gaussian elimination

- Matrices are a convenient way of representing systems of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{cases}$$

- If such a system has a unique solution, it can be solved with Gaussian elimination

Gaussian elimination

- Permitted rules in Gaussian elimination are
  - Rule 1: Interchanging two rows
  - Rule 2: Multiplying a row with a (non-zero) constant
  - Rule 3: Adding a multiple of another row to a row

Gaussian elimination

- Matrices are not necessary for Gaussian elimination, but very convenient, especially augmented matrices
- The augmented matrix corresponding to the previous system of equations is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Gaussian elimination: example

- Suppose we want to solve the following system

$$\begin{cases} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{cases}$$

- Q: What is the geometric interpretation of this system? And what is the interpretation of its solution?

Gaussian elimination: example

- By applying the rules in a clever order, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 2 & 1 & 1 & 15 \\ 1 & 2 & 3 & 26 \end{array}\right] \overset{R3}{\underset{(3)-(1)}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 2 & 1 & 1 & 15 \\ 0 & 1 & 1 & 9 \end{array}\right] \overset{R3}{\underset{(2)-2(1)}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & -1 & -3 & -19 \\ 0 & 1 & 1 & 9 \end{array}\right]$$

$$\overset{R2}{\underset{-1(2)}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 3 & 19 \\ 0 & 1 & 1 & 9 \end{array}\right] \overset{R3}{\underset{(3)-(2)}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & -2 & -10 \end{array}\right] \overset{R3}{\underset{(1)-(2)}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$$\overset{R2}{\underset{-0.5(3)}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & 1 & 5 \end{array}\right] \overset{R3}{\underset{(2)-3(3)}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right] \overset{R3}{\underset{(1)+(3)}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Gaussian elimination: example

- The interpretation of the last augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

is the very convenient system of linear equations

$$\begin{cases} x & = & 3 \\ y & = & 4 \\ z & = & 5 \end{cases}$$

- In other words, the point $(3, 4, 5)$ satisfies all three equations

Gaussian elimination: interpretation

- We started with three equations, which are implicit representations of planes

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

- We ended with three other equations, which can also be interpreted as planes

$$\begin{aligned} x &= 3 \\ y &= 4 \\ z &= 5 \end{aligned}$$

- The steps in Gaussian elimination preserve the location of the solution

Gaussian elimination: outcomes in 3D

- Since any linear equation in three variables is a plane in 3D, we can interpret the possible outcomes of systems of three equations
  1. Three planes intersect in one point: the system has one unique solution
  2. Three planes do not have a common intersection: the system has no solution
  3. Three planes have a line in common: the system has many solutions
- The three planes can also coincide, then the equations are equivalent

Gaussian elimination: inverting matrices

- The same procedure can also be used to invert matrices

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \longmapsto \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\longmapsto \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \longmapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3/2 & -1/2 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 3 & 1 & -1 & 1 \end{array}\right]$$

$$\longmapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3/2 & -1/2 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/3 & -1/3 & 1/3 \end{array}\right] \longmapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/6 & -1/6 & -2/6 \\ 0 & 1 & 0 & -5/6 & 5/6 & -2/6 \\ 0 & 0 & 1 & 2/6 & -2/6 & 2/6 \end{array}\right]$$

Gaussian elimination: inverting matrices

- The last augmented matrix tells us that the inverse of

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

equals

$$\frac{1}{6}\begin{bmatrix} 7 & -1 & -2 \\ -5 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

Gaussian elimination: inverting matrices

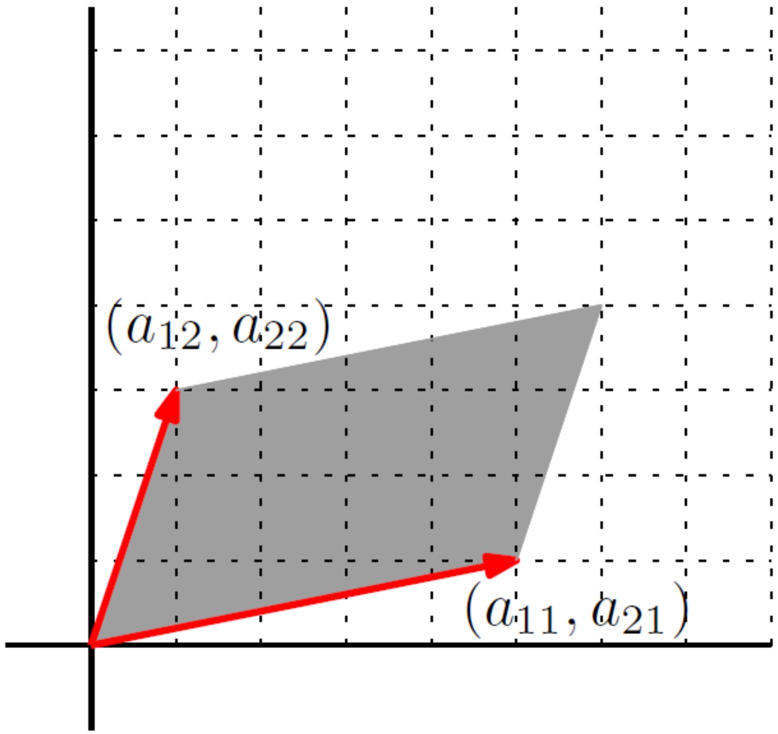
- When does a (square) matrix have an inverse?
- If and only if its columns, seen as vectors, are linearly independent
- Equivalently, if and only if its rows, seen as transposed vectors, are linearly independent

Determinant

- The determinant of a matrix is the signed volume spanned by the column vectors
- The determinant $\det A$ of a matrix A is also written as $|A|$
  - For example

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$



Computing determinant

- Determinant can be computed as follows

  ‘ The determinant of a matrix is the sum of the products of the elements of any row or column of the matrix with their cofactors ’

- But we do not know yet what cofactors are...

Cofactors

- The cofactor of an entry a_{ij} in a $n \times n$ matrix A is the determinant of the $(n-1) \times (n-1)$ matrix A' that is obtained from A by removing the i-th row and the j-th column, multiplied by -1^{i+j}

- We need cofactors to determine the determinant, but we need a determinant to determine a cofactor
  - *' To understand recursion, one needs to understand recursion '*
  - *Q: What is the bottom of this recursion?*

Cofactors

- Example for a 4×4 matrix A, the cofactor of the entry a_{13} is

$$a_{13}^c = \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

and $|A| = a_{13}a_{13}^c - a_{23}a_{23}^c + a_{33}a_{33}^c - a_{43}a_{43}^c$

Determinant and cofactors

- Example

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = 0 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 1 \begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} + 2 \begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}$$

$$= 0(32 - 35) - 1(24 - 30) + 2(21 - 24)$$

$$= 0$$

Systems of equations and determinant

- Consider the following system of linear equations

$$\begin{cases} x + y + 2z & = & 17 \\ 2x + y + z & = & 15 \\ x + 2y + 3z & = & 26 \end{cases}$$

- Such a system of n equations in n unknowns can be solved by using determinants

- In general, if we have $Ax = b$, then $x_i = \frac{|A^i|}{|A|}$ where A^i is obtained from A by replacing the i-th column with b

Systems of equations and determinant

- So for our system $\begin{cases} x+y+2z &= 17 \\ 2x+y+z &= 15 \\ x+2y+3z &= 26 \end{cases}$, we have

$$x = \frac{\begin{vmatrix} 17 & 1 & 2 \\ 15 & 1 & 1 \\ 26 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 1 & 17 & 2 \\ 2 & 15 & 1 \\ 1 & 26 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} 1 & 1 & 17 \\ 2 & 1 & 15 \\ 1 & 2 & 26 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}}$$